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Solitons in an erbium-doped nonlinear fibre medium with stimulated inelastic scattering

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Abstract. We propose the possibility of soliton-type pulse propagation in an optical guide with two-level resonant impurities and high-order effects like: higher-order dispersion, self-steepening, stimulated Raman scattering and stimulated Brillouin scattering. To establish the above results we apply the Painlevé singularity structure analysis and find that the system of equations admits the Painlevé property only for certain values of the physical parameters involved in the system. The Lax pair of the system equation is explicitly shown. As the system without resonant impurities admits solitons, this system is also expected to admit the remarkable solitons property.

In an optical-fibre communication system the major drawbacks are the optical losses and the pulse broadening due to dispersion [1]. To avoid these problems repeaters are placed at regular intervals so that the optical pulses can be reshaped and can be sent without any losses or cross talk. When the distance through which the pulse has to be transmitted is increased the system using repeaters seems to be clumsy and costly. The other way of avoiding this problem is the transmission of soliton pulses. Soliton pulses will not change shape or width in the course of its propagation. This is due to the balance between the dispersion and the nonlinear effect in the form of self-phase modulation [2].

To avoid the problem of optical losses the resonant impurities like erbium atoms are doped to the fibre core. This doping will make the fibre appear to be transparent to the particular wavelength at which the soliton pulse is propagated. This effect is called the self-induced transparency (SIT). In addition to these effects, higher-order effects like higher-order dispersion, self-steepening and stimulated inelastic scattering will also appear in the fibre. So, the exact practical case of the pulse propagation in an erbium-doped nonlinear fibre with higher-order effects is governed by the coupled system of the higher-order nonlinear Schrödinger (HNLS) equation and the Maxwell–Bloch (MB) equations. The main aim of this paper is to show the possibility for soliton-type pulse propagation in the coupled system of HNLS–MB equations for the first time.

Self-phase modulation (SPM) is one of the nonlinear effects which occur in a fibre due to the Kerr effect [2]. Kerr effect can be easily induced in the fibre when the intensity of the pulse is far above a certain threshold value. SPM will produce additional side bands in the frequency component of the pulse. In the negative dispersion regime the effect of SPM is of an opposing nature to that of the dispersion. So, the balance between them will make the pulse travel like solitons. The system equation is governed by the well known nonlinear Schrödinger equation (NLS) which is of the form

$$E_z = \mathrm{i}[\tfrac{1}{2}E_{tt} + |E|^2 E] \qquad (1)$$

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where E is the slowly varying field and the subscripts denotes the partial derivatives. The single soliton of equation (1) is given by [2]

$$E = \mathrm{sech}(t)\exp(\mathrm{i}z/2). \tag{2}$$

According to theory if a pulse of the shape given by equation (2) is transmitted then the pulse will travel as a soliton. However, in practice [3] other problems are faced due to the omission of the higher-order effects in the equation.

Hasegawa [4] has shown that the omitted terms include effects like higher-order dispersion, self-steepening and stimulated inelastic scattering. The above effects are also present within the spectrum of the soliton pulse. Including all the effects the equation appears as

$$E_z = \mathrm{i}\alpha_1[\tfrac{1}{2}E_{tt} + |E|^2E] - \varepsilon[\alpha E_{ttt} + \beta|E|^2E_t + \gamma E(|E|^2)_t] \tag{3}$$

where α, β and γ represent the coefficients related to higher-order dispersion, self-steepening and stimulated inelastic scattering respectively. The presence of higher-order terms have their own physical significance.

The higher-order dispersion effect which is included as the third time derivative of the field also causes the pulse to spread. The other higher-order terms are the nonlinear effects, namely self-steepening and the stimulated inelastic scattering which includes the stimulated Raman scattering (SRS) and the stimulated Brillouin scattering (SBS). Self-steepening will give the pulse a very narrow width in the course of the propagation. This is mainly due to the intensity dependence of the group velocity ($v_g = 1/k'$), where k is the wavenumber. So, the peak of the pulse will travel slower than the wings. The SRS and SBS effects will force the pulse to undergo a frequency shift. This is called the self-frequency shift of the soliton pulse.

So far in the discussion we have not considered the effect due to optical losses. In practice two methods are widely adopted to overcome the problem of attenuation in the fibre. One of the methods is the Raman amplification which is found to be reasonable for this purpose. We still need a periodic coherent light system to give the pump pulses for the amplification and another drawback of Raman amplification is that it is also a nonlinear process; this means that it needs a very high intensity beam. Another method to overcome this problem is by the utilization of fibres doped with two-level resonant impurities like erbium atoms.

Consider a non-dispersive, two-level resonant impurity-doped linear dielectric light guide. The pulse propagation in this system is governed by the MB equations [5]. The MB equations have the structure

$$\begin{aligned} E_z &= \langle p \rangle \\ p_t &= 2\mathrm{i}\omega p + 2E\eta \\ \eta_t &= -(Ep^* + E^*p) \end{aligned} \tag{4}$$

where p, η are given by $v_1 v_2^*$ and $|v_1|^2 - |v_2|^2$ respectively. Here v_1 and v_2 are the wavefunctions of the two energy levels of the doped impurities. The bracketed term $\langle \ldots \rangle$ is the averaging function over the entire frequency range:

$$\langle p(z,t;\omega) \rangle = \int_{-\infty}^{\infty} p(z,t;\omega)g(\omega)\,\mathrm{d}\omega \qquad \int_{-\infty}^{\infty} g(\omega)\,\mathrm{d}\omega = 1. \tag{5}$$

The pulse propagation in the MB equations is also found to be a soliton-type pulse propagation [6]. The presence of the resonant atom makes the system transparent to the particular wavelength of the pulse. So, the system becomes transparent to pulse propagation induced by the pulse itself and, hence, it is called SIT.

If a fibre doped with erbium is considered, the system equation for the more general case will be the coupled system of the HNLS equation and MB equations. The possibility of the co-existence of SIT and NLS solitons is discussed in many references [7–11]. When effects like higher-order dispersion and self-steepening are included then the NLS equations will be replaced by Hirota's equation. In [12] we proposed a coupled system of Hirota's equation and the MB equations; the single-soliton solution is also explicitly shown. Here we propose, for the first time, a coupled system of the HNLS equation and the MB equations. Hereafter we refer to this system as the HNLS–MB system. The HNLS–MB system is found to have the form

$$
\begin{aligned}
E_z &= \mathrm{i}\alpha_1[\tfrac{1}{2}E_{tt} + |E|^2E] - \varepsilon[\alpha E_{ttt} + \beta|E|^2E_t + \gamma E(|E|^2)_t] + 2\alpha_2\langle p\rangle \\
p_t &= 2\mathrm{i}\omega p + 2E\eta \\
\eta_t &= -(Ep^* + E^*p).
\end{aligned}
\tag{6}
$$

We now have to check the integrability condition of the HNLS–MB equations; the Painlevé analysis [13, 14] of equation (6) has been carried out. For convenience, the function $g(\omega)$ in equation (5) is considered as a Dirac delta function at the resonant frequency so that the averaging function $\langle p\rangle$ in equation (6) will be replaced by p itself. Also, the independent variables z and t are interchanged. So, the set of HNLS-MB equations whose Painlevé analysis is going to be analysed is:

$$
\begin{aligned}
E_t &= \mathrm{i}\alpha_1[\tfrac{1}{2}E_{zz} + |E|^2E] - \varepsilon[\alpha E_{zzz} + \beta|E|^2E_z + \gamma E(|E|^2)_z] + 2\alpha_2 p \\
p_z &= 2\mathrm{i}\omega p + 2E\eta \\
\eta_z &= -(Ep^* + E^*p).
\end{aligned}
\tag{7}
$$

As E and P are complex and η is real for the analysis, let us express $E = a$, $E^* = b$, $p = c$, $p^* = d$ and $\eta = e$, to give

$$
\begin{aligned}
a_t &= \mathrm{i}\alpha_1[\tfrac{1}{2}a_{zz} + a^2b] - \varepsilon[\alpha a_{zzz} + (\beta+\gamma)aba_z + \gamma a^2b_z] + 2\alpha_2 c \\
b_t &= -\mathrm{i}\alpha_1[\tfrac{1}{2}b_{zz} + b^2a] - \varepsilon[\alpha b_{zzz} + (\beta+\gamma)bab_z + \gamma b^2a_z] + 2\alpha_2 d \\
c_z &= 2\mathrm{i}\omega c + 2ae \\
d_z &= -2\mathrm{i}\omega d + 2be \\
e_z &= -(ad + bc).
\end{aligned}
\tag{8}
$$

The generalized Laurent expansions of $a, \ldots, e$ are

$$
\begin{aligned}
a &= \varphi^l \sum_{j=0}^{\infty} a_j(z,t)\varphi^j(z,t) \\
b &= \varphi^m \sum_{j=0}^{\infty} b_j(z,t)\varphi^j(z,t) \\
c &= \varphi^n \sum_{j=0}^{\infty} c_j(z,t)\varphi^j(z,t) \\
d &= \varphi^o \sum_{j=0}^{\infty} d_j(z,t)\varphi^j(z,t) \\
e &= \varphi^q \sum_{j=0}^{\infty} e_j(z,t)\varphi^j(z,t)
\end{aligned}
\tag{9}
$$

with $a_0, \ldots, e_0 \neq 0$, where l, m, n, o and q are negative integers and $a_j, \ldots, e_j$ and φ are the set of expansion coefficients which are analytic in the neighbourhood of the non-characteristic singular manifold $\varphi(z,t) = z + \psi(t) = 0$. Looking at the leading order, we substitute $a \approx a_0\varphi^l, \ldots, e \approx e_0\varphi^q$ in equation (8), and on balancing the dominant terms we obtain

$$
l = m = -1 \qquad n = o = q \qquad a_0 b_0 = -1 \qquad b_0 c_0 = a_0 d_0. \tag{10}
$$

Substituting the full expansion of the Laurent series and keeping the leading-order terms only, we obtain

$$
\begin{vmatrix}
A & Ba_0^2 & 0 & 0 & 0 \\
Bb_0^2 & A & 0 & 0 & 0 \\
-2e_0 & 0 & (j+n) & 0 & -2a_0 \\
0 & -2e_0 & 0 & (j+n) & -2b_0 \\
d_0 & c_0 & b_0 & a_0 & (j+n)
\end{vmatrix} = 0 \tag{11}
$$

where $A = \alpha(j-1)(j-2)(j-3) + (j-1)(\beta+\gamma)a_0b_0 - (\beta+3\gamma)a_0b_0$ and $B = (j-1)-(\beta+\gamma)$. Substituting equation (10) into equation (11) and solving the determinant, the resonance values are found to be

$$
j = -1, 0, 3, 4, -n, \pm\left[\frac{24\alpha}{\beta+2\gamma}\right]^{1/2} - n, 3 \pm 2\left[\frac{\beta-\gamma}{\beta+2\gamma}\right]^{1/2}. \tag{12}
$$

From this careful analysis we find that equation (12) admits a sufficient number of positive resonances when $n = -2$ and for the condition $3\alpha = \beta = 2\gamma$ only; this gives emphasis to the fact that equation (8) is non-integrable for other values of α, β and γ. Substituting the values of n, α, β and γ, the resonance values are found to be

$$
j = -1, 0, 0, 2, 2, 3, 4, 4, 4. \tag{13}
$$

As usual, the resonance value at $j = -1$ corresponds to the arbitrariness of the singularity manifold φ, and $j = 0, 0$ implies that a_0 or b_0, and c_0 or d_0 are arbitrary which is evident from equation (10). Similarly, from the coefficients of $(\varphi^{-3}, \varphi^{-3}, \varphi^{-2}, \varphi^{-2}, \varphi^{-2})$ we obtain

$$\begin{bmatrix} \varepsilon b_0(\beta+3\gamma) & \varepsilon a_0(\beta+\gamma) & 0 & 0 & 0 \\ \varepsilon b_0(\beta+\gamma) & \varepsilon a_0(\beta+3\gamma) & 0 & 0 & 0 \\ 2e_0 & 0 & 1 & 0 & 2a_0 \\ 0 & 2e_0 & 0 & 1 & 2b_0 \\ d_0 & c_0 & b_0 & a_0 & -1 \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \\ e_1 \end{bmatrix} = \begin{bmatrix} -\mathrm{i}\alpha(1+a_0b_0) \\ \mathrm{i}\alpha(1+a_0b_0) \\ -2\mathrm{i}c_0 \\ 2\mathrm{i}d_0 \\ 0 \end{bmatrix}. \tag{14}$$

Solving equation (14), the values of $a_1, \ldots, e_1$ are found to be

$$\begin{aligned} a_1 &= -\mathrm{i}\alpha(1+a_0b_0)/(2b_0\gamma) \\ b_1 &= \mathrm{i}\alpha(1+a_0b_0)/(2a_0\gamma) \\ c_1 &= 2\mathrm{i}a_0e_0 \\ d_1 &= -2\mathrm{i}b_0e_0 \\ e_1 &= 2\mathrm{i}(b_0c_0 - a_0d_0)/3 = 0. \end{aligned} \tag{15}$$

Similarly, we also obtain the matrix for $a_2, \ldots, e_2$ as

$$\begin{bmatrix} 2\varepsilon\gamma a_0b_0 & \varepsilon\beta a_0^2 & 0 & 0 & 0 \\ \varepsilon\beta b_0^2 & 2\varepsilon\gamma a_0b_0 & 0 & 0 & 0 \\ 2e_0 & 0 & 0 & 0 & 2a_0 \\ 0 & 2e_0 & 0 & 0 & 2b_0 \\ d_0 & c_0 & b_0 & a_0 & 0 \end{bmatrix} \begin{bmatrix} a_2 \\ b_2 \\ c_2 \\ d_2 \\ e_2 \end{bmatrix} = \begin{bmatrix} D \\ F \\ -2\mathrm{i}c_1 - 2a_1e_1 \\ 2\mathrm{i}d_1 - 2b_1e_1 \\ -a_1d_1 - b_1c_1 \end{bmatrix} \tag{16}$$

where

$$D = -a_0\psi_t - \mathrm{i}\alpha_1(2a_0b_0a_1 + a_0^2b_1) - \varepsilon\beta a_0a_1b_1 - \varepsilon\gamma(a_0a_1b_1 + a_1^2b_0) - 2\alpha_2c_0$$

and

$$F = -b_0\psi_t + \mathrm{i}\alpha_1(2a_0b_0b_1 + b_0^2a_1) - \varepsilon\beta b_0b_1a_1 - \varepsilon\gamma(b_0b_1a_1 + b_1a_0) - 2\alpha_2d_0.$$

Equation (16) reveals that there are two sources of arbitrariness, corresponding to the resonances $j = 2, 2$. From the remaining powers of φ we find that equation (8) admits a sufficient number of arbitrary functions for $3\alpha = \beta = 2\gamma$ only. Hence, we conclude that equations (6) are expected to be integrable for the above choice of parameters only. The integrable version of equation (6) takes the form

$$\begin{aligned} E_z &= \mathrm{i}\alpha_1[\tfrac{1}{2}E_{tt} + |E|^2E] - \varepsilon[E_{ttt} + 3|E|^2E_t + \tfrac{3}{2}E(|E|^2)_t] + 2\alpha_2 p \\ p_t &= 2\mathrm{i}\omega p + 2E\eta \\ \eta_t &= -(Ep^* + E^*p). \end{aligned} \tag{17}$$

As system (17) admits the Painlevé property we believe that one can also establish the integrability for the general case, i.e. with $\langle p \rangle$, and construct an auto Bäcklund transformation, Lax pair and soliton solutions by truncating the Laurent series at the constant-level terms:

$$\begin{aligned} a &= a_0\varphi^{-1} + a_1 \\ b &= b_0\varphi^{-1} + b_1 \\ c &= c_0\varphi^{-2} - c_1\varphi^{-1} + c_2 \\ d &= d_0\varphi^{-2} + d_1\varphi^{-1} + d_2 \\ e &= e_0\varphi^{-2} + e_1\varphi^{-1} + e_2. \end{aligned} \tag{18}$$

However, it should be noted that because of the complicated mathematical structure of the system equations it is very difficult to construct the Lax pair from the Painlevé analysis point of view. Based on the earlier investigations, we are now able to construct the Lax pair of the HNLS–MB equations. The linear eigenvalue problem of the system equation is found to be

$$\psi_t = \mathbf{U}\psi \qquad \psi = (\psi_1\psi_2\psi_3)^{\mathrm{T}} \tag{19}$$

$$\psi_z = \mathbf{V}\psi \tag{20}$$

where

$$\mathbf{U} = \begin{pmatrix} -\mathrm{i}\lambda & 0 & E\mathrm{e}^{-\mathrm{i}\theta} \\ 0 & -\mathrm{i}\lambda & E^*\mathrm{e}^{\mathrm{i}\theta} \\ -E^*\mathrm{e}^{\mathrm{i}\theta} & -E\mathrm{e}^{-\mathrm{i}\theta} & \mathrm{i}\lambda \end{pmatrix}$$

$$\begin{aligned} \mathbf{V} = {}& -4\mathrm{i}\varepsilon\lambda^3 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} + 4\varepsilon(\lambda^2 - |E|^2) \begin{pmatrix} 0 & 0 & E\mathrm{e}^{-\mathrm{i}\theta} \\ 0 & 0 & E^*\mathrm{e}^{\mathrm{i}\theta} \\ -E^*\mathrm{e}^{\mathrm{i}\theta} & -E\mathrm{e}^{-\mathrm{i}\theta} & 0 \end{pmatrix} \\ & + 2\mathrm{i}\varepsilon\lambda \begin{pmatrix} |E|^2 & (E\mathrm{e}^{-\mathrm{i}\theta})^2 & (E\mathrm{e}^{-\mathrm{i}\theta})_x \\ (E^*\mathrm{e}^{\mathrm{i}\theta})^2 & |E|^2 & (E^*\mathrm{e}^{\mathrm{i}\theta})_x \\ (E^*\mathrm{e}^{\mathrm{i}\theta})_x & (E\mathrm{e}^{-\mathrm{i}\theta})_x & -2|E|^2 \end{pmatrix} \\ & - \varepsilon \begin{pmatrix} 0 & 0 & (E\mathrm{e}^{-\mathrm{i}\theta})_{xx} \\ 0 & 0 & (E^*\mathrm{e}^{\mathrm{i}\theta})_{xx} \\ -(E^*\mathrm{e}^{\mathrm{i}\theta})_{xx} & -(E\mathrm{e}^{-\mathrm{i}\theta})_{xx} & 0 \end{pmatrix} \\ & + \varepsilon((EE_x^* - E_xE^*) + (\mathrm{i}/3\varepsilon)|E|^2) \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ & + \mathrm{i}\alpha_2/(\lambda - \mathrm{i}\omega) \begin{pmatrix} \eta & 0 & -p\mathrm{e}^{-\mathrm{i}\theta} \\ 0 & -\eta & p^*\mathrm{e}^{\mathrm{i}\theta} \\ -p^*\mathrm{e}^{\mathrm{i}\theta} & p\mathrm{e}^{-\mathrm{i}\theta} & 0 \end{pmatrix}. \end{aligned}$$

Here $\theta = (1/6\varepsilon)[z - (t/18\varepsilon)]$, $x = z - (t/12\varepsilon)$ and λ is the spectral parameter. The consistency condition $\mathbf{U}_z - \mathbf{V}_t + [\mathbf{U}, \mathbf{V}] = 0$ leads to equation (17) with appropriate co-ordinate transformations.

The Lax pair of equation (17) confirm its complete integrability. As the HNLS–MB equations without the MB part have the soliton solutions given in [15], we expect the same type of soliton solution and also the splitting of the initial one-soliton pulse shape for certain parameters. (For further details of the HNLS soliton see [15].) The system equation is found to be integrable for the condition derived from the Painlevé analysis only; a similar condition also prevails for the HNLS equation [15]. This is because of the balancing between the physical parameters α (higher-order dispersion), β (self-steepening) and γ (stimulated inelastic scattering) and the erbium-doped fibre support soliton-type pulse propagation, with the balancing between the pulse-spread due to higher-order dispersion, and the spectral spreading due to the self-steepening and the stimulated inelastic scattering. It is an interesting problem to show the above behaviours for system (17) and to see the possible experimental evidence; at present we are investigating these aspects and the results will be given elsewhere.

Thus, in this paper we have proposed a new type of coupled higher-order nonlinear Schrödinger equation with the Maxwell–Bloch equations, which describes the wave dynamics of the erbium-doped fibre with higher-order nonlinear effects. The proposed system equation is found to allow soliton-type propagation for certain conditions between the parameters involving the higher-order dispersion, self-steepening and the stimulated inelastic scattering only. The Lax pair of the HNLS-MB equations is explicitly shown. So, we conclude that the erbium-doped nonlinear fibre with higher-order nonlinear effects support soliton-type propagation with the complete integrability of the system equation.

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